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Miscellanea

A general rotation method for orthogonal Latin hypercubes

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Summary

Orthogonal Latin hypercubes provide a class of useful designs for computer experiments. Among the available methods for constructing such designs, the method of rotation is particularly prominent due to its theoretical appeal as well as its space-filling properties. This paper presents a general method of rotation for constructing orthogonal Latin hypercubes, making the rotation idea applicable to many more situations than the original method allows. In addition to general theoretical results, many new orthogonal Latin hypercubes are obtained and tabulated.

Some key words: Computer experiment; Orthogonal array; Space-filling design.

1. Introduction

Orthogonal Latin hypercubes have long been recognized as a class of useful designs for computer experiments and numerical integration. This goes back to Iman & Conover (1982), and subsequently Owen (1994) and Tang (1998), who considered nearly-orthogonal Latin hypercubes. Ye (1998) was the first to study exactly orthogonal Latin hypercubes. Since then, many authors have contributed to this research field. Some significant developments include Steinberg & Lin (2006), Lin et al. (2009), Pang et al. (2009), Sun et al. (2009), Lin et al. (2010), Georgiou & Efthimiou (2014), and Sun & Tang (2017).

Among the many methods of constructing orthogonal Latin hypercubes, the method of rotation stands out because of its theoretical elegance. It was introduced in Steinberg & Lin (2006) and further studied by Lin et al. (2009) and Pang et al. (2009). It enjoys some very attractive space-filling properties (Sun & Tang, 2017), but can be criticized for its severe run-size restriction. For example, when two-level orthogonal arrays are to be rotated into orthogonal Latin hypercubes, their run sizes must be of the form 2^{2^u}, which equals 4, 16, 256, 65 536 for $u = 1, 2, 3, 4$.

In this paper we propose and study a general method of rotation, thus rendering the rotation idea applicable to many more scenarios than the original method allows. In fact our general method fills all the run-size gaps left by the original rotation method. In particular, it can be used to rotate two-level orthogonal arrays of 32, 64 and 128 runs into orthogonal Latin hypercubes. We present some general theoretical results as well as some new orthogonal Latin hypercubes that can be constructed by the general method.

A Latin hypercube is an $n \times m$ matrix where each column is a permutation of the levels taken from $\Omega(n) = \{j - (n-1)/2 : j = 0, 1, \ldots, n-1\}$, and it becomes an orthogonal Latin hypercube if any two columns are orthogonal. We use $\mathrm{OLH}(n, m)$ to denote such an orthogonal Latin hypercube of n runs for m

factors. Our use of centred levels in $\Omega(n)$ is to facilitate the study of orthogonality. An orthogonal array of strength t with n runs for m factors is an $n \times m$ matrix with entries from a set of s levels such that in each subarray of t columns, all the t-tuples of levels occur with the same frequency. We use OA(n, m, s, t) to denote such an array. In this paper, the s levels where s is a prime power are taken from a Galois field $\mathrm{GF}(s) = \{\alpha_0 = 0, \alpha_1, \ldots, \alpha_{s-1}\}$, which simplifies to $\mathrm{GF}(s) = \{0, 1, \ldots, s-1\}$ if s is a prime. When an orthogonal array is to be rotated into an orthogonal Latin hypercube, before the rotation takes place, we should convert its levels into equally spaced and centred levels in $\Omega(s) = \{j-(s-1)/2 : j = 0, 1, \ldots, s-1\}$, which can be done by simply replacing α_j by $j - (s-1)/2$ for $j = 0, 1, \ldots, s-1$. When an orthogonal array A that has levels from $\mathrm{GF}(s)$ is made to have levels from $\Omega(s)$, we write the resulting array as A^*.

2. TWO EXAMPLES

As the general method is a bit technical, we present two examples in this section to illustrate the main idea.

Example 1. We give a construction of an OLH$(32, 24)$. From Steinberg & Lin (2006), an OA$(16, 12, 2, 2)$, say A, can be obtained that has the form $A = (A_1, A_2, A_3)$ where each A_j is a full 2^4 factorial. Now define

$$B_j = \begin{pmatrix} A_j & A_j \\ A_j & 1 + A_j \end{pmatrix}$$

where $1 + A_j$ is the matrix obtained by adding 1, mod 2, to all the entries of A_j. In fact, B_j is simply the double of A_j (Chen & Cheng, 2006), and it takes a slightly different form here because we are using 0 and 1 instead of ± 1 to denote the two levels. Now let

$$B_j = (B_{j1}, B_{j2}), \quad B_{j1} = \begin{pmatrix} A_j \\ A_j \end{pmatrix}, \quad B_{j2} = \begin{pmatrix} A_j \\ 1 + A_j \end{pmatrix}.$$

A moment of thought reveals that a subarray of five columns, given by (B_{j1}, b) for any column b of B_{j2} or by (b, B_{j2}) for any column b of B_{j1}, gives a complete 2^5 factorial design. Let $s = 2$ and consider

$$R = \begin{pmatrix} s^4 & -s^3 & -s^2 & s & -1 & 0 & 0 & 0 \\ s^3 & s^4 & -s & -s^2 & 0 & 1 & 0 & 0 \\ s^2 & -s & s^4 & -s^3 & 0 & 0 & 1 & 0 \\ s & s^2 & s^3 & s^4 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & s^4 & -s^3 & -s^2 & s \\ 0 & -1 & 0 & 0 & s^3 & s^4 & -s & -s^2 \\ 0 & 0 & -1 & 0 & s^2 & -s & s^4 & -s^3 \\ 0 & 0 & 0 & 1 & s & s^2 & s^3 & s^4 \end{pmatrix}.$$

Clearly, the columns of matrix R are mutually orthogonal and the nonzero entries in each column are a signed permutation of 1, 2, 4, 8, 16. This, in conjunction with the aforementioned property of B_j, implies that $C_j = B_j^* R$ is an OLH$(32, 8)$, where $B_j^* = B_j - 0{\cdot}5$ simply converts two levels 0 and 1 into two centred levels $\pm 0{\cdot}5$. Finally, we obtain an OLH$(32, 24)$ by taking $C = (C_1, C_2, C_3)$.

Example 2. We construct an OLH$(27, 8)$. Let A be an OA$(9, 4, 3, 2)$ and write $A = (A_1, A_2)$ where A_1 and A_2 each have two columns. Let

$$B_j = (B_{j1}, B_{j2}), \quad B_{j1} = \begin{pmatrix} A_j \\ A_j \\ A_j \end{pmatrix}, \quad B_{j2} = \begin{pmatrix} A_j \\ 1 + A_j \\ 2 + A_j \end{pmatrix}$$

where the calculations for $1 + A_j$ and $2 + A_j$ are both modulo 3. It can be easily checked that a subarray of three columns, given by (B_{j1}, b) for any column b of B_{j2} or by (b, B_{j2}) for any column b of B_{j1}, is a complete 3^3 factorial. This shows that $C_j = B_j^* R$ where $B_j^* = B_j - 1$ is an OLH(27, 4) and hence (C_1, C_2) is an OLH(27, 8), and where

$$R = \begin{pmatrix} 9 & -3 & -1 & 0 \\ 3 & 9 & 0 & 1 \\ 1 & 0 & 9 & -3 \\ 0 & -1 & 3 & 9 \end{pmatrix}$$

has mutually orthogonal columns and the nonzero entries in each column are a signed permutation of 1, 3, 9.

3. General results

3·1. *A general class of rotation matrices*

Let

$$R_{10} = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}, \quad R_{u0} = \begin{pmatrix} s^{2^{(u-1)}} R_{(u-1)0} & -R_{(u-1)0} \\ R_{(u-1)0} & s^{2^{(u-1)}} R_{(u-1)0} \end{pmatrix} \quad (u = 2, 3, \ldots).$$

These are the rotation matrices used in Steinberg & Lin (2006) and Pang et al. (2009) for constructing orthogonal Latin hypercubes. More specifically, R_{u0} is a matrix of order 2^u, where the columns are mutually orthogonal and the entries in each column are a signed permutation of $1, s, \ldots, s^{2^u-1}$. We will build a sequence of rotation matrices based on each R_{u0}. First let

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_u = \begin{pmatrix} Q_{u-1} & 0 \\ 0 & -Q_{u-1} \end{pmatrix} \quad (u \geqslant 2).$$

Then define

$$R_{u1} = \begin{pmatrix} sR_{u0} & -Q_u \\ Q_u & sR_{u0} \end{pmatrix}, \quad R_{uv} = \begin{pmatrix} sR_{u(v-1)} & -Q_{u+v-1} \\ Q_{u+v-1} & sR_{u(v-1)} \end{pmatrix} \quad (v \geqslant 2). \tag{1}$$

Clearly, R_{uv} in (1) is of order 2^{u+v}. We have obtained a sequence of rotation matrices $R_{u0}, R_{u1}, R_{u2}, \ldots$ based on each R_{u0}. This is supported by Lemma 1.

LEMMA 1. (i) *The columns of* R_{uv} *are mutually orthogonal.*
(ii) *The nonzero entries in each of its columns are a signed permutation of* $1, s, s^2, \ldots, s^{2^u+v-1}$.

The proofs of Lemma 1 and all later results are deferred to the Appendix.

3·2. *Main results*

To construct orthogonal Latin hypercubes using the rotation matrices R_{uv} in (1), we first construct an orthogonal array whose columns are grouped into subarrays of 2^{u+v} columns, each of which has the property that the subset of columns corresponding to the $2^u + v$ nonzero entries in each column of R_{uv} is a full factorial. Lemmas 2 and 3 are the foundation of our construction method. We recall that the levels of orthogonal arrays are from $\mathrm{GF}(s) = \{\alpha_0 = 0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}\}$.

LEMMA 2. *If* $(e_1, \ldots, e_g, f)$ *is an s-level full factorial, so is* $(e_1, \ldots, e_g, f + e_j)$ *for any* e_j.

Let $w = (\alpha_0, \alpha_1, \ldots, \alpha_{s-1})^{\mathrm{T}}$, and define $d_j = \alpha_j w$ for $j = 0, 1, \ldots, s-1$. Matrix $(d_0, d_1, \ldots, d_{s-1})$ is in fact a difference scheme. One can regard Lemma 3 below as a high-dimensional variant of Lemma 6·27 of Hedayat et al. (1999).

LEMMA 3. *Let A be an s-level full factorial for g factors and let $B_j = d_j \oplus A$, the Kronecker sum of d_j and A. Then (B_i, b) for any b in B_j must be an s-level full factorial in $g+1$ factors for any $i \neq j$.*

We are now ready to present a construction of orthogonal Latin hypercubes motivated by the general rotation matrices R_{uv} in (1). Let $B^{(0)}$ be a full factorial of 2^u factors of s levels. Create

$$B_{j_1} = \left(d_{2j_1} \oplus B^{(0)},\ d_{2j_1+1} \oplus B^{(0)}\right), \quad j_1 = 0, 1, \ldots, [s/2]-1,$$

where $[x]$ denotes the greatest integer not exceeding x. In total, $[s/2]$ such B_{j_1} can be created, each of which, according to Lemma 3, has the property that certain subarrays of $2^u + 1$ factors are full factorials, which allows an orthogonal Latin hypercube to be generated via $B^*_{j_1} R_{u1}$ where $B^*_{j_1}$ is obtained from B_{j_1} by replacing level α_j by level $j - (s-1)/2$. Now, for each B_{j_1}, create

$$B_{j_1 j_2} = \left(d_{2j_2} \oplus B_{j_1},\ d_{2j_2+1} \oplus B_{j_1}\right), \quad j_2 = 0, 1, \ldots, [s/2]-1.$$

In total, there are $[s/2]^2$ such $B_{j_1 j_2}$, each of which, by applying Lemma 3 twice, has the property that certain subarrays of $2^u + 2$ factors are full factorials, thus making $B^*_{j_1 j_2} R_{u2}$ an orthogonal Latin hypercube. In general, define

$$B_{j_1 \cdots j_v} = \left(d_{2j_v} \oplus B_{j_1 \cdots j_{v-1}},\ d_{2j_v+1} \oplus B_{j_1 \cdots j_{v-1}}\right), \quad j_v = 0, 1, \ldots, [s/2]-1.$$

Each of these $[s/2]^v$ $B_{j_1 \cdots j_v}$ has a structure that allows itself to be rotated into an orthogonal Latin hypercube via $B^*_{j_1 \cdots j_v} R_{uv}$. Combining the columns of these designs obtained from all the $B_{j_1 \cdots j_v}$, we obtain an orthogonal Latin hypercube with $m = [s/2]^v\, 2^{u+v}$ columns. Now suppose $A = (A_1, \ldots, A_k)$ is an orthogonal array such that each A_j is a full factorial of 2^u factors of s levels. If we use each A_j as the $B^{(0)}$ in the above construction and then combine all the columns of orthogonal Latin hypercubes obtained from all the A_j, we obtain an orthogonal Latin hypercube with $m = k[s/2]^v\, 2^{u+v}$ columns. We summarize these developments as a theorem.

THEOREM 1. *Let $A = (A_1, \ldots, A_k)$ be an s-level orthogonal array of strength 2 such that each A_j is a full factorial of 2^u factors. Then using the rotation matrix R_{uv}, an* OLH(n, m) *can be constructed with $n = s^{2^u+v}$ runs and $m = k[s/2]^v\, 2^{u+v}$ factors.*

Theorem 1 generalizes the results of Steinberg & Lin (2006) and Pang et al. (2009), which correspond to the special case of $v = 0$ in Theorem 1, and thus allows designs with much more flexible run sizes to be generated. By taking $v = 1, 2, \ldots, 2^u$, we see that Theorem 1 fills all the gaps in the run sizes left by the original rotation method. Though Theorem 1 is applicable to any $v \geqslant 0$, the existence of $A = (A_1, \ldots, A_k)$ in Theorem 1 requires that $k2^u(s-1) \leqslant s^{2^u} - 1$. Construction of such arrays A was discussed earlier in Steinberg & Lin (2006) and Pang et al. (2009), and recently by Sun & Tang (2017).

Example 3. The OLH$(32, 24)$ obtained in Example 1 corresponds to $u = 2$, $s = 2$ and $v = 1$ applied to the array A there. If we continue to apply our method, we obtain an OLH$(64, 48)$ from taking $v = 2$ and an OLH$(128, 96)$ from $v = 3$.

If an OLH(s, m') is available, then an orthogonal Latin hypercube with more columns can be constructed using the idea of Lin et al. (2009). This is done as follows. As in Theorem 1, let $A = (A_1, \ldots, A_k)$ be an orthogonal array of strength 2 with each A_j being a full factorial of 2^u factors. Then each of the m' columns of an OLH(s, m') can be used to generate one copy of A. Rotating these m' copies of A separately using R_{uv} and then combining all the columns, we obtain the next result.

COROLLARY 1. *If an* $\mathrm{OLH}(s, m')$ *is available, then an* $\mathrm{OLH}(n, m)$ *can be constructed that has* $n = s^{2^u+v}$ *runs and* $m = m'k[s/2]^v\ 2^{u+v}$ *factors.*

Example 4. Let us start with an $\mathrm{OA}(25, 6, 5, 2)$, say $A = (A_1, A_2, A_3)$. Because $s = 5$, we have $[s/2] = 2$. Thus we can obtain an $\mathrm{OLH}(125, 24)$ by Theorem 1. Since an $\mathrm{OLH}(5, 2)$ is available, Corollary 1 gives an $\mathrm{OLH}(125, 48)$.

3·3. *Further results*

Let us take a closer look at the problem of constructing $\mathrm{OLH}(s^3, m)$ by the method of rotation. This requires the use of R_{11} as in (1) and the construction of an orthogonal array A of s^3 runs whose columns can be arranged in the form $A = (A_1, \ldots, A_k)$, where each $A_i = (A_{i1}, A_{i2})$ with both A_{i1} and A_{i2} having two columns, and with the property that (A_{i1}, a) for any column a of A_{i2} and (a, A_{i2}) for any column a of A_{i1} are of strength 3. But this property of $A_i = (A_{i1}, A_{i2})$ is actually equivalent to that of A_i itself being of strength 3. In § 3·2, we construct such an array A from an orthogonal array of s^2 using a difference scheme. In this section, we provide a direct solution to the problem.

We want to construct an orthogonal array $\mathrm{OA}(s^3, m, s, 2)$ that has the form $A = (A_1, \ldots, A_k)$ such that each A_j is an array of strength 3 for four factors. We will do this by carefully choosing sets of four columns from the saturated $\mathrm{OA}(s^3,\ s^2{+}s{+}1, s, 2)$ obtained by the Rao–Hamming construction.

Let a, b, c be the three independent columns, i.e., (a, b, c) forms a full factorial in three factors of s levels. Then all the columns in the $\mathrm{OA}(s^3,\ s^2{+}s{+}1, s, 2)$ can be represented by $\beta_1 a + \beta_2 b + \beta_3 c$ with the first nonzero β set to 1, where $\beta_i \in \mathrm{GF}(s)$.

Define $C_{i0} = (a{+}\alpha_i c,\ b{+}\alpha_i c)$ for $i = 0, 1, \ldots, s-1$. For $j = 1, 2, \ldots, [(s-1)/2]$, define $C_{ij} = (a{+}\alpha_{2j-1} b{+}\alpha_i c,\ a{+}\alpha_{2j} b{+}\alpha_i c)$ where $i = 0, 1, \ldots, s-1$.

LEMMA 4. *We have:*
(i) $(C_{ij}, C_{i'j})$ *has strength* 3 *for any* j *and* $i \neq i'$*;*
(ii) $(C_{(s-1)0}, C_{01})$ *has strength* 3 *so long as* $\alpha_1 \neq -1$ *and* $\alpha_2 \neq -1$*;*
(iii) $(C_{(s-1)j}, C_{0(j+1)})$ *has strength* 3 *for any* $j \geqslant 1$.

The condition for part (ii) of Lemma 4 cannot hold if $s \leqslant 3$. But when $s > 3$, we can always relabel the elements of $\mathrm{GF}(s)$ to make $\alpha_1 \neq -1$ and $\alpha_2 \neq -1$. We will give a separate treatment for $s = 3$ later in the section. The following general result works for any $s > 3$.

Create the following list of ordered C_{ij}:

$$C_{00}, \ldots, C_{(s-1)0}, C_{01}, \ldots, C_{(s-1)1}, C_{02}, \ldots, C_{(s-1)2}, \ldots.$$

By Lemma 4, any two adjacent C_{ij} taken together from the above list is an array of strength 3. If we take two C_{ij} at a time from the above list to form A_1, A_2 and so on, we then obtain

$$A = (A_1, \ldots, A_k) \quad \text{where each } A_j \text{ is an } \mathrm{OA}(s^3, 4, s, 3), \tag{2}$$

with

$$k = \left\{\frac{1}{2}\left(s + s\left[\frac{s-1}{2}\right]\right)\right\}. \tag{3}$$

Now in each A_j, we replace level α_j by $j - (s-1)/2$ to obtain A_j^*. Then $(A_1^* R_{11}, \ldots, A_k^* R_{11})$ is an orthogonal Latin hypercube for $m = 4k$ factors.

THEOREM 2. *For any prime power* $s \geqslant 3$*, an* $\mathrm{OLH}(s^3, m)$ *can be constructed where*

$$m = \begin{cases} s^2, & s \text{ even}, \\ s^2 + s - 2, & s = 4q + 1 \text{ for some integer } q, \\ s^2 + s, & s = 4q + 3 \text{ for some integer } q. \end{cases}$$

Table 1. *Some orthogonal Latin hypercubes,* OLH(n, m)

s	n	m	Rotation matrix	Source
2	32	24	R_{21}	Theorem 1 and Example 3
2	64	48	R_{22}	Theorem 1 and Example 3
2	128	96	R_{23}	Theorem 1 and Example 3
2	512	496	R_{31}	Theorem 1
3	27	12	R_{11}	Theorem 2 and Example 5
3	243	80	R_{21}	Theorem 1
5	125	58	R_{11}	Corollary 2 and Example 6
7	343	168	R_{11}	Corollary 2 and Example 6
9	729	445	R_{11}	Corollary 2 and Example 6

For $s > 3$, Theorem 2 is established by the preceding general construction, as one can easily verify that the number $m = 4k$ of factors with k as in (3) has the form stated in the theorem. For $s = 3$, we construct (A_1, A_2, A_3) directly with $A_1 = (a, b, c, abc)$, $A_2 = (ab, ab^2, ac, ac^2)$ and $A_3 = (bc, bc^2, abc^2, ab^2c^2)$ where $a^{\beta_1}b^{\beta_2}c^{\beta_3}$ is a shortcut notation for $\beta_1 a + \beta_2 b + \beta_3 c$.

An application of Theorem 1 to the case of $u = 1$ and $v = 1$ produces an OLH(s^3, m) with the number m of factors given by $m = 4[(s+1)/2][s/2]$, which equals s^2 for even s and $s^2 - 1$ for odd s. We see that Theorem 2 is capable of constructing an OLH(s^3, m) with a larger m whenever s is odd.

In obtaining $A = (A_1, \ldots, A_k)$ as in (2) by choosing columns from the saturated OA$(s^3, s^2+s+1, s, 2)$, not all columns are selected. For even s, the $s+1$ leftover columns are actually on the same line, in the language of projective geometry, and no three of them can have strength 3. For $s = 4q+3$, there is only one leftover column. For $s = 4q+1$, the three leftover columns are actually independent. Together, they can be used to create one more orthogonal column of s^3 levels.

PROPOSITION 1. *An* OLH(s^3, s^2+s-1) *can be constructed where* $s = 4q+1$ *for some* q *is a prime power.*

Example 5. For $s = 3$ and $s = 7$, Theorem 2 gives an OLH$(27, 12)$ and an OLH$(343, 56)$. For $s = 5$ and $s = 9$, Proposition 1 gives an OLH$(125, 29)$ and an OLH$(729, 89)$. In contrast, Theorem 1 gives an OLH$(27, 8)$, an OLH$(125, 24)$, an OLH$(343, 48)$ and an OLH$(729, 80)$ for $s = 3, 5, 7$ and 9, respectively.

Similar to Corollary 1, combining the idea of Lin et al. (2009) with Theorem 2 and Propostion 1, we obtain the next result provided that an OLH(s, m') is available.

COROLLARY 2. *If an* OLH(s, m') *exists, then an* OLH(s^3, m) *can be constructed where*

$$m = \begin{cases} m's^2, & s \text{ even}, \\ m'(s^2+s-1), & s = 4q+1 \text{ for some integer } q, \\ m'(s^2+s), & s = 4q+3 \text{ for some integer } q. \end{cases}$$

Example 6. For $s = 5, 7$ and 9, using the existing OLH$(5, 2)$, OLH$(7, 3)$ and OLH$(9, 5)$ see (Lin et al. 2009), Corollary 2 then gives an OLH$(125, 58)$, an OLH$(343, 168)$, and an OLH$(729, 445)$.

Finally, we present in Table 1 a collection of orthogonal Latin hypercubes that can be constructed using our methods. Some of these have been discussed in the early examples. All designs in Table 1 are new.

4. DISCUSSION

Through the introduction of a rich class of rotation matrices, we have developed a method of constructing orthogonal Latin hypercubes that fills all the gaps in run sizes left by the original rotation method. The construction requires the creation of an orthogonal array whose sets of columns corresponding to the nonzero entries of a rotation matrix are full factorials. While the method of constructing such arrays in § 3·2 using difference schemes is more generally applicable, our direct method for OA$(s^3, m, s, 2)$ in § 3·3 proves to be more powerful. It would be interesting to investigate whether a direct construction will still work for the case of OA$(s^5, m, s, 2)$. Steinberg & Lin (2006) and Sun & Tang (2017) showed that the rotation method enjoys some attractive space-filling properties. The orthogonal Latin hypercubes given by Theorem 1, Theorem 2 and Proposition 1 are all OA-based Latin hypercubes that achieve stratifications on $s \times s$ grids in two dimensions, but another future research direction would be to determine whether these designs also possess some stronger space-filling properties (Sun & Tang, 2017). It would be also interesting to examine how to use our general rotation method to construct Latin hypercubes with higher-order orthogonal properties. A result from Sun et al. (2011, Theorem 1) should be useful in this regard.

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Appendix

Proof of Lemma 1

By a simple induction argument, we can show that $R_{uv}^{\mathrm{T}} Q_{u+v} = Q_{u+v} R_{uv}$, from which it follows that R_{uv} is a matrix with both its rows and its columns mutually orthogonal. This proves Lemma 1(i). Lemma 1(ii) is obvious.

Proof of Lemma 2

Let $e_{1h}, \ldots, e_{gh}$ and f_h be the hth elements of $e_1, \ldots, e_g$ and f, respectively. Then for any $\beta_1, \ldots, \beta_g$ and β_{g+1} in GF(s) and any h, we must have that

$$e_{1h}=\beta_1, \ldots, e_{gh}=\beta_g,\ f_h+e_{jh}=\beta_{g+1} \quad \text{if and only if} \quad e_{1h}=\beta_1, \ldots, e_{gh}=\beta_g,\ f_h=\beta_{g+1}-\beta_j.$$

Since $\left(e_1, \ldots, e_g, f\right)$ is a full factorial, Lemma 2 is immediate.

Proof of Lemma 3

Simply let $B_i = (e_1, \ldots, e_g)$ and $f = (d_j - d_i) \oplus 0_{sg}$. Then B_j can be written as $(e_1 + f, \ldots, e_g + f)$. Then Lemma 3 follows from Lemma 2.

Proof of Lemma 4

We only give proofs for parts (i) and (ii). The proof for part (iii) is similar to that for part (i).

We now prove Lemma 4(i). For $j = 0$, $(C_{ij}, C_{i'j})$ becomes $(C_{i0}, C_{i'0}) = (a+\alpha_i c,\ b+\alpha_i c,\ a+\alpha_{i'} c,\ b+\alpha_{i'} c)$, which will have strength 3 if we show that none of its columns can be written as a linear combination of two other columns. For example, assume that $a + \alpha_{i'} c = \beta_1(a + \alpha_i c) + \beta_2(b + \alpha_i c)$ for some β_1 and β_2 from GF(s). Since a, b, c are independent, we must have $\beta_1 = 1$, $\beta_2 = 0$ and $\alpha_{i'} = \beta_1 \alpha_i + \beta_2 \alpha_i$, which leads to $\alpha_{i'} = \alpha_i$, contradicting $i \neq i'$. The proof for $j \geqslant 1$ is very similar.

To prove Lemma 4(ii), consider the four columns $a + \alpha_{s-1} c,\ b + \alpha_{s-1} c,\ a + \alpha_1 b,\ a + \alpha_2 b$ given by $(C_{(s-1)0}, C_{01})$. Clearly, $a + \alpha_{s-1} c$ or $b + \alpha_{s-1} c$ cannot be linear combinations of $a + \alpha_1 b$ and $a + \alpha_2 b$. Now suppose that $a + \alpha_1 b = \beta_1(a + \alpha_{s-1} c) + \beta_2(b + \alpha_{s-1} c)$ for some $\beta_1, \beta_2 \in$ GF(s). Then we must have $\beta_1 = 1$, $\beta_2 = \alpha_1$ and $\beta_1 + \beta_2 = 0$, which implies that $\beta_1 + \beta_2 = 1 + \alpha_1 = 0$. This is impossible since

$\alpha_1 \neq -1$. Similarly, we can show that $a + \alpha_2 b$ cannot be a linear combination of $a + \alpha_{s-1} c$ and $b + \alpha_{s-1} c$ if $\alpha_2 \neq -1$.

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